Using Counter-Examples to Enhance Learners' Understanding of Undergraduate Mathematics

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Introduction

The suggested practice is based on the usage of counter-examples as a pedagogical strategy. It can improve students' conceptual understanding in mathematics, reduce their common misconceptions, provide a broader view on the subject and enhance students' critical thinking skills. The suggested practice is one of numerous pedagogical strategies. It effectiveness depends a lot on the lecturer's enthusiasm towards it. Another support factor is sharing the practice with colleagues through seminars, publications, conferences, etc. In my opinion there are no organisational barriers to the practice. The practice is generic. It can be used in all subjects and disciplines. It is part of our ability to think and analyse. The practice can improve students' critical thinking skills that can be successfully used outside the university. Many students exposed to the practice commented in formal questionnaires that it helped them to understand concepts better, prevent mistakes in future, eliminate some misconceptions, expand their example space, develop logical and critical thinking, and made their participation in lectures more active.

1. What are Counter-Examples?

In the information age analysing given information and making a quick decision on whether it is true or false is an important ability. A counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule) is false. It only takes one counter-example to disprove a statement. Counter-examples play an important role in mathematics and other subjects. They are a powerful and effective tool for scientists, researchers and practitioners. They are good indicators that show a suggested hypothesis or chosen direction of research is wrong. Before trying to prove the conjecture or hypothesis it is often worth looking for a possible counter-example – doing so can save much of time and effort.

Counter-examples also provide an important means of communicating ideas in mathematics, which in itself may be viewed as making conjectures and then either proving or disproving them by counter-examples. Here are a couple of well-known cases to illustrate the point:

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1. For a long time mathematicians tried to find a formula for prime numbers. The numbers of the form 2^(2^n) + 1 , where *n* is natural, were once considered as prime numbers, until a counter-example was found. For *n* = 5 that number is composite: 2^(2^5) + 1 = 4294967297 = 641 x 6700417.
2. Another conjecture about prime numbers is still waiting to be proved or disproved – Goldbach's or the Goldbach-Euler conjecture, posed by Goldbach in his letter to Euler in 1742. It looks deceptively simple at first. It states that *every even number greater than 2 is the sum of 2 prime numbers*. For example, 12 = 5 + 7, 20 = 3 + 17, and so on. A powerful computer was used in 1999 to search for counter-examples to that conjecture. No counter-examples have been found up to 4×10^{14} . In 2000 the publishers Faber & Faber offered a US$1 million prize to anyone who could prove or disprove that conjecture. To date the prize remains unclaimed.

2. Benefits to Learners

The intention of this publication is to encourage teachers and students to use counter-examples in the teaching/learning of mathematics to:

- deepen conceptual understanding
- reduce or eliminate common misconceptions
- advance students' mathematical understanding beyond the merely procedural or algorithmic
- enhance critical thinking skills – analyzing, justifying, verifying, checking, proving – to benefit students in other areas of their lives
- expand students' "example set" of functions and bring to light connections and relationships among mathematical ideas
- make learning more active and creative.

1. To deepen conceptual understanding

Many students nowadays are used to concentrating on techniques, manipulations and familiar procedures and pay little attention to the concepts, conditions of the theorems, properties of the functions, and to reasoning and justification.
'When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions (Mason & Watson, 2001). Paying attention to the conditions of theorems, for example, can help engineering students develop the good habit of considering the extreme conditions to which new devices will be subjected. Aircraft are designed to fly in storms and turbulence, not just in perfect weather! The ability to pay attention to the conditions of a sale offer is essential in everyday life. We all know the importance of reading the fine print, checking the 'special conditions apply'.

A recent case study (Klymchuk, 2005) showed that the use of counter-examples in teaching improved the students' performance on test questions that required conceptual understanding.

2. To reduce or eliminate misconceptions

Over recent years, partly due to extensive use of modern technology, the proof component of the traditional approach in teaching Calculus (definition–theorem–proof–example–application) has almost disappeared. Students are used to relying on technology and sometimes lack logical thinking and conceptual understanding. At times Calculus courses are taught in such a way that special cases are avoided and students are exposed only to 'nice' functions and 'good' examples, especially at school level. This approach can create many misconceptions that can be explained by Tall's generic extension principle: 'If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts' (Tall, 1991).

There is a difference between students' misconceptions in basic Algebra and in Calculus. There are no textbooks where 'properties' like $(a + b)^2 = a^2 + b^2$ can be found, and nobody teaches such 'rules'. Some introductory Calculus textbooks, on the other hand, especially school textbooks, contain incorrect statements. For example: "*If the graph of a function is a continuous and smooth curve (no sharp corners) on (a,b), then the function is differentiable on (a,b)*", and *"a tangent line to a curve is a line that just touches the curve at one point and does not cross it there"*. Some students actually *learn* Calculus this way. Practice in creating counter-examples can help students reduce or eliminate such misconceptions before they become second nature.

3. To advance students' mathematical understanding

Creating examples and counter-examples is neither algorithmic nor procedural and requires advanced mathematical thinking which is not often taught at school. 'Coming up with examples requires different cognitive skills from carrying out algorithms – one needs to look at mathematical objects in terms of their properties. To be asked for an example can be disconcerting. Students have no pre-learned algorithms to show the "correct way" (Selden & Selden, 1998). Practice in constructing their own examples and counter-examples can help students enhance their creativity and advance their mathematical thinking.

4. To enhance critical thinking skills

Creating counter-examples to wrong statements has a distinct advantage over constructing examples of functions satisfying certain conditions, because counter-examples deal with disproving, justification, argumentation, reasoning and critical thinking – the essence of mathematical thinking. These skills will benefit students not only in their university study but also in other areas of life that have nothing to do with mathematics.

5. To expand a student's "example set"

After creating or being exposed to many functions with interesting properties, students will expand their "example set", allowing them to better communicate their ideas in mathematics and in practical applications. While creating counter-examples students learn a lot about the behaviour of functions and can later apply this knowledge to solving real-life problems. As Henry Pollak from Bell Laboratories, USA pointed out "the society provides time for mathematics to be taught in schools, colleges and universities not because mathematics is beautiful, which it is, or because it provides a great training for the mind, but because it is so useful" (Pollak, 1969).

6. To make learning more active and creative

Teaching experience shows that the use of counter-examples as a pedagogical strategy can create a discovery learning environment and make learning more active.

In 2003 my colleague and I conducted an international study of more than 600 students from 10 universities in different countries looking at their attitudes to using counter-examples in teaching/learning mathematics (Gruenwald & Klymchuk, 2003). The study showed that the students' attitudes were very positive – 92% of the participants indicated that the pedagogical strategy was very effective. They reported it helped them understand concepts better, prevented mistakes, developed logical and critical thinking, and made learning mathematics more challenging, interesting and creative. Some common students' comments:

- helps me to think and question deeply; gives more sound knowledge of the subject
- we can understand more; it makes me think more effectively
- can prevent mistakes
- you gain a better understanding
- it makes the problem clearer
- it boosts self-confidence
- it helps you retain information that you have learned
- it is a good teaching tool
- it teaches you to question everything
- it makes you think carefully about the concepts and how they are applied
- it makes you think critically
- it supports self-control
- it requires logical thinking, not only calculations
- it makes problems more understandable
- it is hard but it is fun
- it is a good way to select top students
- I can look at maths from another angle
- it is good not only in mathematics
- it really forces you to think hard
- it is not a routine exercise, it is creative.

In 2005 a formal survey of 54 students at AUT University revealed that for 52 of them (96%) using counter-examples helped them learn.

3. How to Use Counter-Examples in Mathematics Teaching

There are different ways of using counter-examples in teaching:

- giving the students a mixture of correct and incorrect statements
- asking students to create their own wrong statements and counter-examples to them
- making a deliberate mistake in the lecture
- asking the students to spot an error on a certain page of their textbook
- giving the students bonus marks towards their final grade for providing excellent counter-examples to hard questions during the lecture
- including questions that require constructing counter-examples into assignments and tests

Here we will look at some ways counter-examples can be used as a pedagogical strategy in teaching/learning of Calculus.

Dealing with counter-examples for the first time can be challenging for many students. Some might not see the difference between "proving" a correct statement by an example and disproving a wrong statement by a counter-example. They know that it is impossible to prove a theorem by a certain example, but it is hard for them to accept that one can disprove a statement by a single counter-example. Sometimes they think that a particular counter-example is just an exception to the rule and there are no other ‘pathological’ cases. “Students quite often fail to see a single counter-example as disproving a conjecture. This can happen when a counter-example is perceived as ‘the only one that exists’, rather than being seen as generic” (Selden & Selden, 1998).

With some experience, students become comfortable with and interested in creating counter-examples. The activity of using counter-examples to disprove wrong statements can generate many questions for discussion, for example:

- How can you change the statement to make it correct?
- What can you change in the given function and still have it as a counter-example?
- Which other wrong statements can your particular counter-example refute?
- Can you use another type of function as a counter-example?
- Can you construct the most general class of counter-examples?

It is important that students learn to pay attention to every single detail of a statement – the word order, the symbols used (for example, the shape of the brackets defining an open and a closed interval), the locality in the statement (either at a certain point or on an interval) and so on. For example, the following statement is correct:

> “if a function $f(x)$ if differentiable on (a,b) and its derivative is positive for all x from (a,b), then the function is increasing on (a,b)”.

The two statements below look very similar to the above statement but both are incorrect:

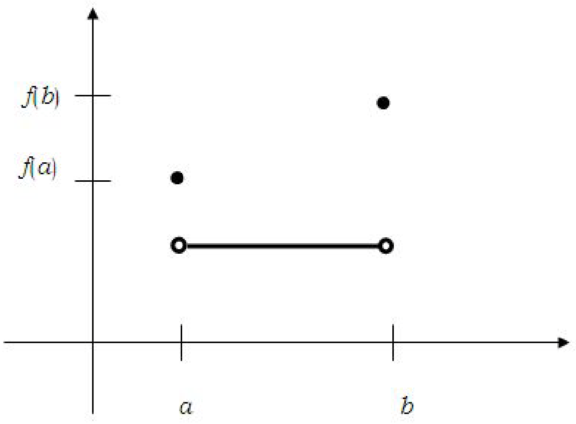
> “if a function $f(x)$ is differentiable on (a,b) and its derivative is positive at a point $x = c$ from (a,b), then there is a neighbourhood of the point $x = c$ where the function is increasing”;

> “if a function $f(x)$ if differentiable on its domain and its derivative is positive for all x from its domain, then the function is increasing everywhere on its domain”.

Below are some notes from my experience using counter-examples with students. The following statement is considered as an example:

Statement: If a function $y = f(x)$ is defined on $[a,b]$ and continuous on (a,b), then for any N between $f(a)$ and $f(b)$ there is some point c between a and b such that $f(c) = N$.

The only difference between this statement and the Intermediate Value Theorem is the shape of the brackets of the interval where the function is continuous, that is continuity of the function is required on an open interval (a,b), instead of a closed interval $[a,b]$. When students are asked to disprove the statement they usually come up with something like this:

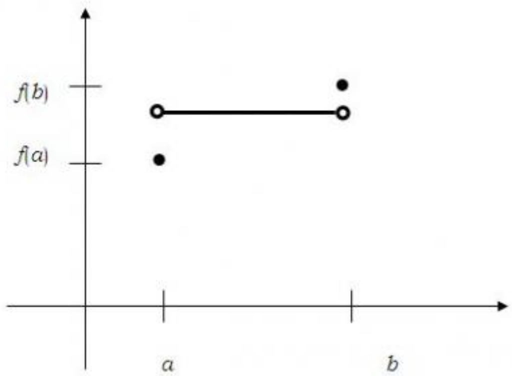


To generate discussion and create other counter-examples one can suggest that:

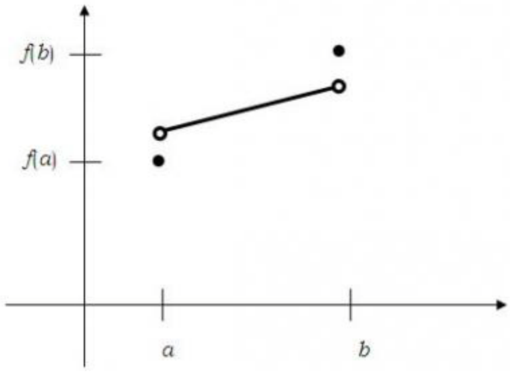
In the above graph the statement's conclusion is not true for any value of N between f(a) and f(b). Modify the graph in such a way that the statement's conclusion is true for:
a) one value of N between f(a) and f(b)
b) infinitely many but not all values of N between f(a) and f(b).

One can expect the students to produce the following sketches:

a)



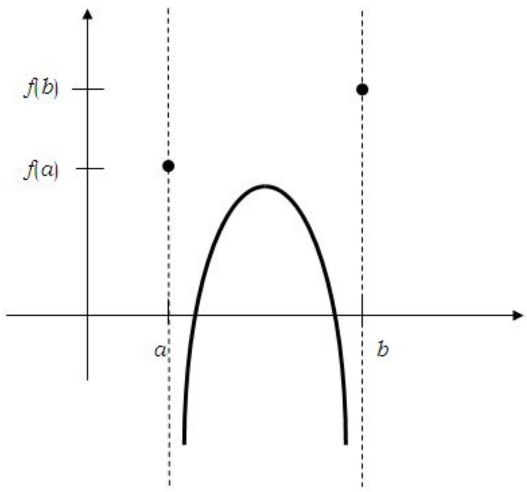
b)



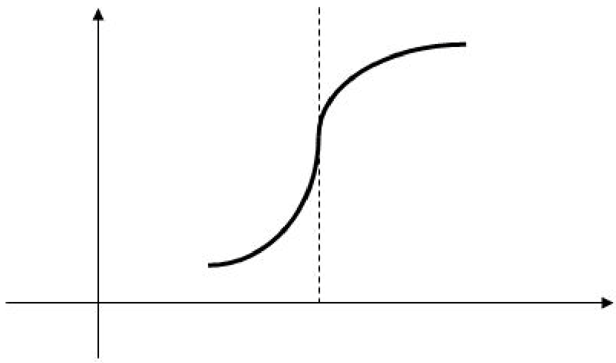
Another challenge can be presented:

Give a counter-example such that the graph doesn't have white circles.

In this case students may come up with something like this:

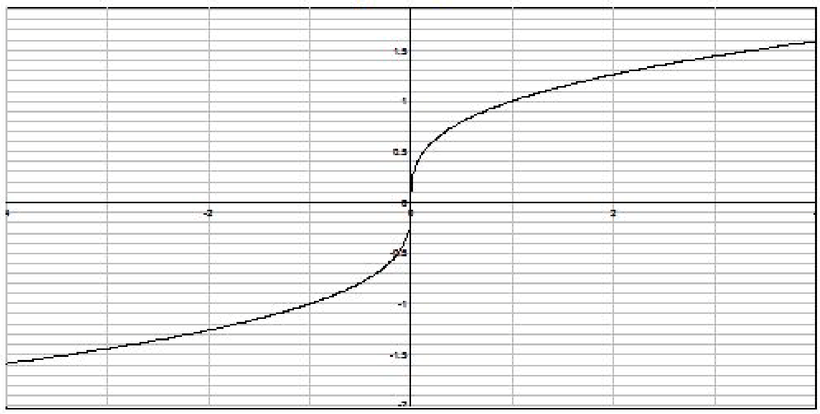


Finally, some tips on how to search for possible counter-examples. In many cases a simple sketch of the graph is enough to create a counter-example so the formula is not needed. For example, to disprove the statement "*if a function is continuous on the interval (a,b) and its graph is a smooth curve (no sharp corners) on that interval, then the function is differentiable at any point on (a,b)*", one can give a simple sketch of a smooth curve that has a vertical tangent at a certain point:



Alternatively, the graph of the cube root function along with the formula can be presented:

$y = \sqrt[3]{x}$



To create counter-examples it is useful to remember the graphs of functions with interesting properties. Depending on the statements to be disproved, one can sketch the graphs of piecewise functions, in particular step functions, the graphs of functions with sharp corners, with cusps, with oscillation, with damping factors and so on. The graphs of such functions are presented in most books on Calculus. A good selection of colourful, well illustrated graphs of functions with interesting behaviour that illustrate a variety of concepts from Calculus can be found in (Stewart, 2001) and (Smith & Minton, 2002). More exotic functions include the functions that are defined in one way for rational numbers and another way for irrational numbers, for example the Dirichlet function. The Dirichlet like functions are not continuous at any point of their domain. They are called nowhere continuous functions or everywhere discontinuous functions. A good introduction to the Dirichlet function is given in (Dunham, 2005) on p.197.

When searching for a suitable graph for a particular counter-example, comparing it with the graphs suggested by peer students and their lecturer students can expand their ‘example set’. Such a collection of interesting graphs will definitely improve students’ understanding and communicating concepts and contexts from Calculus.

So, get ready to use counter-examples as a pedagogical strategy and have fun disproving!

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